Synthesis of Single Qutrit Circuits from Clifford + R gates

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Outline

- **Motivation**
- Problem
- Numerical result
- Outlook

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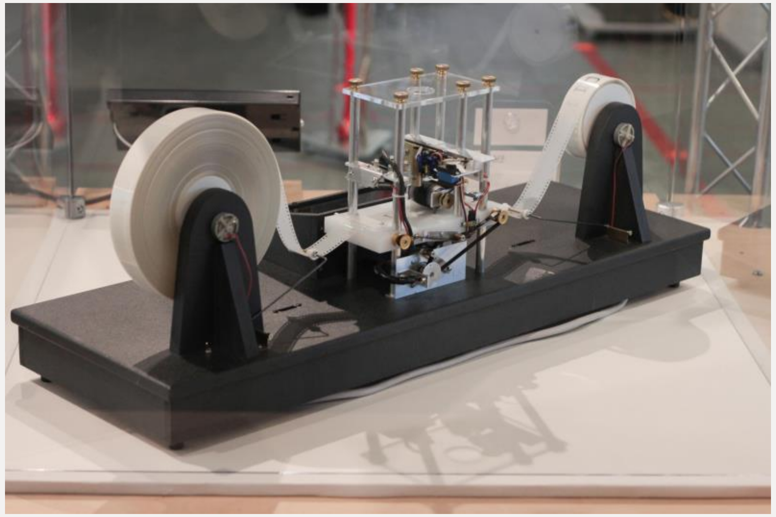
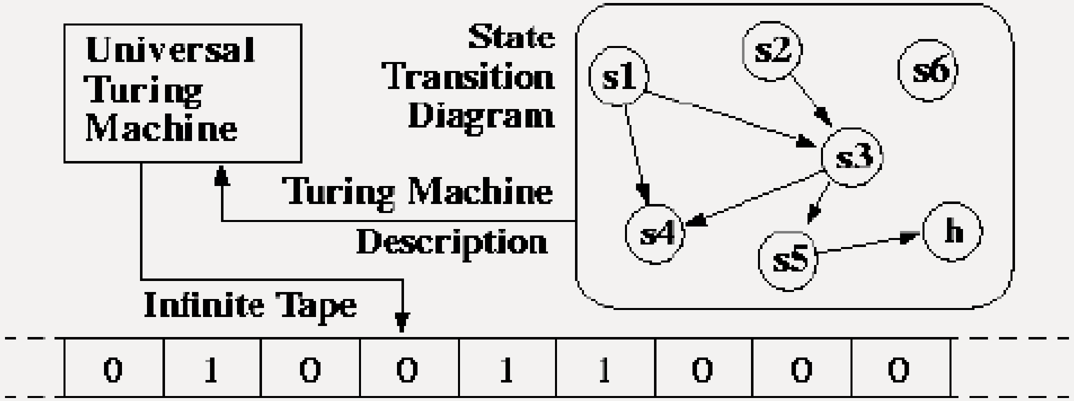
Classical Turing machine





The Turing machine has:

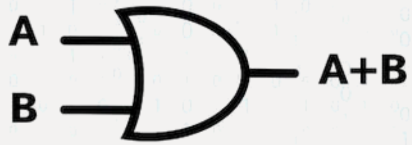
- Infinite tape
- Read/Write head
- State register
- Transition function

Turing machine can:

- Solve any computable problem

Universal classical computation

- A computer is **universally computational** if it can perform any algorithmic task that a Turing machine can

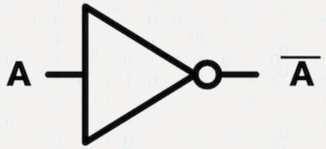


2 input OR gate

A	B	A+B
0	0	0
0	1	1
1	0	1
1	1	1

Modern computers using circuit models are **universal**, with universal gatesets:

- Single **NAND** gate
- Single **NOR** gate
- **AND**, **OR**, and **NOT** gates



2 input NOT gate

A	$\overline{A}$
0	1
1	0

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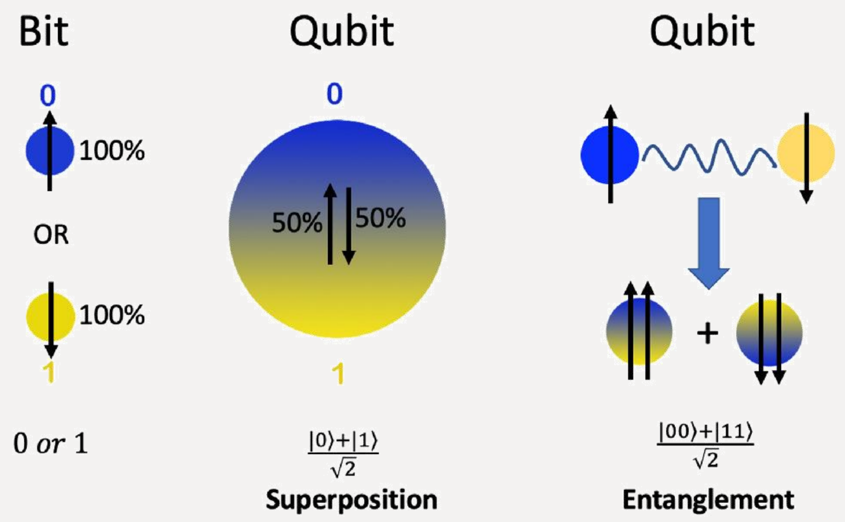
Quantum computing

- The quantum computing extends the classical machine by allowing:
  - **Superposition**
  - **Unitary evolution**
  - **Entanglement**



- Quantum circuit models allow for **universal quantum computation** if it has:
  - Any single-qubit unitary operation (infinitely many, impossible!)
  - One entangling gate

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Quantum gate set

- A quantum gate set is universal if it satisfies:
  - Any single-qubit unitary operation can be approximated (relaxed condition)
  - At least one entangling gate

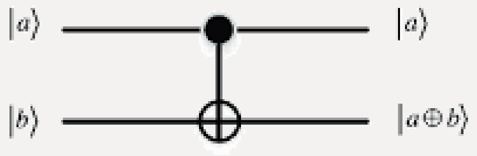
Solovay-Kitaev Theorem (approximate unitary operations): If a gate set generates a dense subgroup of the group SU(2), then any unitary in SU(2) can be approximated efficiently with a sequence of gates from the gate set.

- Universal gatesets:
  - Clifford + T includes {H, S, CNOT, T}
  - {H, CNOT, T}
  - {H, CCNOT}

  Works because <H, T> is dense in SU(2)

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

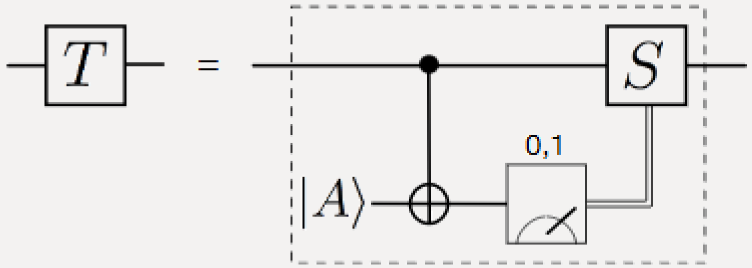


$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Non-Clifford gate T is expensive

T gate is expensive in fault-tolerant quantum computing:

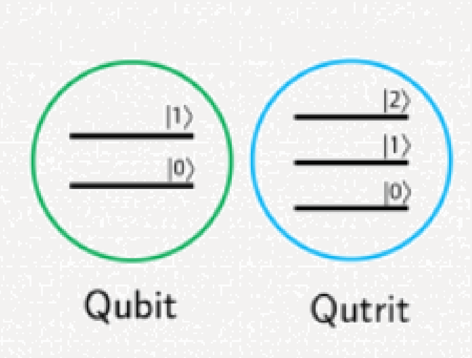
- It cannot be implemented transversally for most quantum error correction codes being studied
- Requires resource-intensive techniques, such as magical state distillation



Gottesman-Knill Theorem (Clifford gates alone not enough): Quantum computations using only Clifford gates and stabilizer states can be efficiently simulated by a classical computer.

The addition of a non-Clifford gate (such as a T gate) is necessary to achieve quantum universality.

Qutrit



A three-level quantum system, superposition of three basis states

- Stores more information
- Fewer gates for some operations
- Better error correction threshold
- More efficient magic state distillation [QIP 2025]

But

- Gate designs are more complex and less standardized
- Qutrit hardware at the initial phase

We study single qutrit gate synthesis using a particular gate set called **Clifford + R** that includes {H, S, R}

Clifford:

$$H = \frac{1}{i\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$S = \mathrm{Diag}(1, \omega, 1)$$

Non-Clifford: $\mathbf{R} = \mathrm{Diag}(1, 1, -1)$

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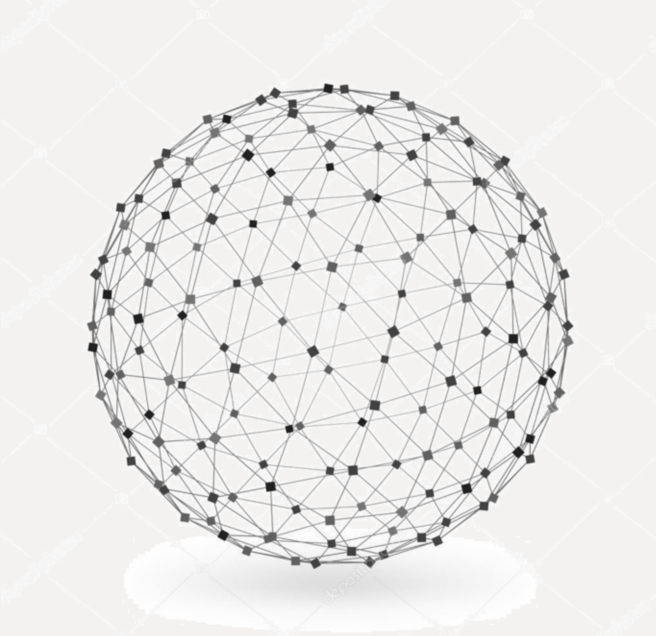
Problem statement

Goal: Given a unitary matrix $R^{Z}_{(0,1)}(\theta) = Diag(e^{-\frac{i\theta}{2}}, e^{\frac{i\theta}{2}}, 1)$ and target error ϵ, find a unitary approximation V that can be exactly synthesized from Clifford + R, such that $\left\| V - R^{Z}_{(0,1)}(\theta) \right\| \leq \epsilon$.

It is sufficient for all single-qutrit gate synthesis:

- 8 Givens rotations $R^{P}_{(a,b)}(\theta)$ generate the full SU(3)
- Remaining rotations can be obtained from Clifford transformation



Two steps

1. Find V over some integer ring to **ϵ-approximate** $R^{Z}_{(0,1)}(\theta)$
2. Exact **synthesis** V using Clifford + R gateset

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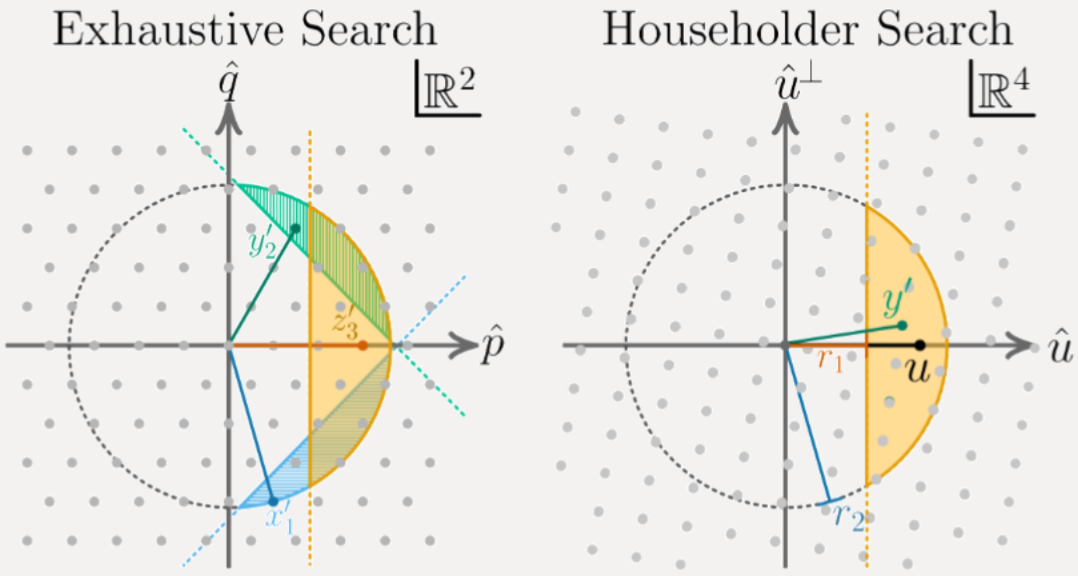
Step 1: ϵ-approximation (find V)

Two algorithms:

- Exhaustive search over Clifford + R
- Seach over the Householder reflections



- Lower bound: $N_{\mathbf{R}}(\varepsilon) \geq 3\log_3 \frac{1}{\varepsilon} + C$
- $N_{\mathbf{R}}(\varepsilon) = 4.11\log_3(1/\varepsilon) + C$
- runtime: $O\left(\varepsilon^{-4.4}\right)$

- Lower bound: $N_{\mathbf{R}}(\varepsilon) \geq 5\log_3 \frac{1}{\varepsilon} + C$
- $N_{\mathbf{R}}(\varepsilon) = 5.12\log_3(1/\varepsilon) + C$
- runtime: $O\left(\varepsilon^{-0.4}\right)$

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Step 2: exact synthesis V

The approximation V can be written in a unique normal form:

$$V = \prod_{i}^{f} HD(a_{0,i}, a_{1,i}, a_{2,i})\mathbf{R}^{\varepsilon_i} X^{\delta_i}$$

Iteratively search over the parameters

Algorithm 3: Decomposition of V in $U(3, \mathcal{R}_{3,\chi})$.

Data: Unitary U in $U(3, \mathcal{R}_{3,\chi})$.
T_D - table of all zero-sde unitaries in $U(3, \mathcal{R}_{3,\chi})$.
Result: Sequence S_{out} of $H, D, \mathbf{R}$, and X gates that implement U.

$u \leftarrow$ top left entry of U
$S_{\text{out}} \leftarrow$ Empty
$s \leftarrow \text{sde}(u)$
while $s > 0$ **do**
  state $\leftarrow$ unfound
  forall $a_0, a_1, a_2, \delta \in \{0, 1, 2\}, \varepsilon \in \{0, 1\}$ **do**
    while *state* $=$ *unfound* **do**
      $u \leftarrow (HD(a_0, a_1, a_2)\mathbf{R}^{\varepsilon} X^{\delta} U)_{00}$
      if $\text{sde}(u) = s - 1$ **then**
        state $\leftarrow$ found
        append $X^{-\delta}\mathbf{R}^{-\varepsilon}D^{-1}H$ to S_{out}
        $s \leftarrow \text{sde}(u)$
        $U \leftarrow HD(a_0, a_1, a_2)\mathbf{R}^{\varepsilon} X^{\delta} U$
lookup matrix S_{rem} for U in T_D
append S_{rem} to S_{out}
return S_{out}

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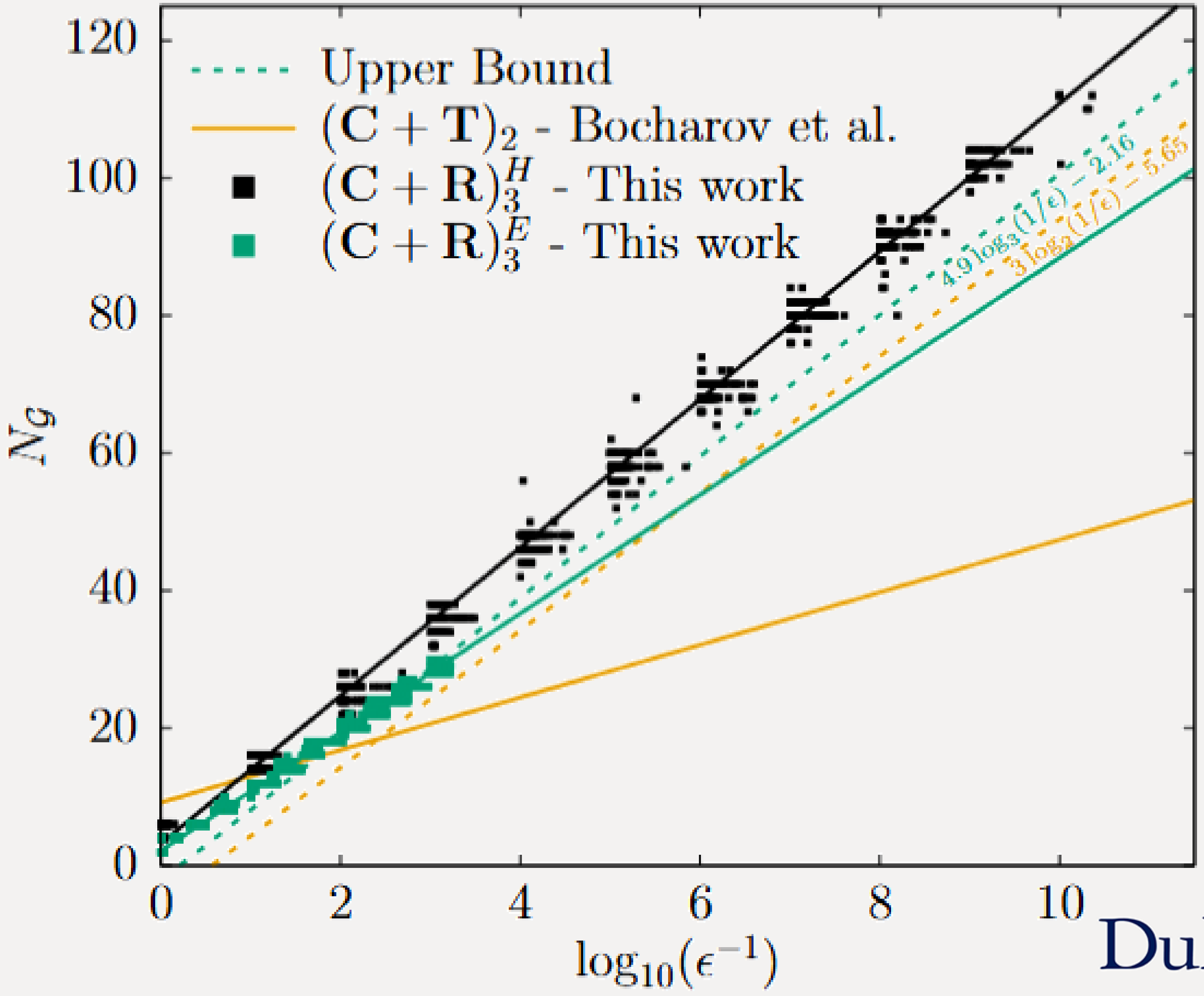
Numerical result

Number of non-Clifford gates vs infidelity



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Outlook

- Synthesize with Clifford + D, or Clifford + T
- Look for larger transverse groups other than Clifford
- Synthesis gate other than $R^{Z}_{(0,1)}(\theta)$

Thank you!

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