EXAM 1

Math 212, 2019 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)*

(a) Find the area of the parallelogram defined by $\vec{v} = (1, 2, 0)$ and $\vec{w} = (3, 0, 4)$.

$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 2 & 0 \\ 3 & 0 & 4 \end{pmatrix} = (8, -4, -6)$$

$$\text{area} = \|\vec{v} \times \vec{w}\| = \sqrt{8^2 + (-4)^2 + (-6)^2} = \sqrt{116} = 2\sqrt{29}$$

(b) Compute the angle between $\vec{v}$ and $\vec{w}$.

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right)$$

$$= \arccos\left(\frac{3}{\sqrt{5}\sqrt{25}}\right)$$

$$= \arccos\left(\frac{3}{5\sqrt{5}}\right)$$

(c) The plane $-8x + 4y + 6z = 5$ passes through a point whose position vector is $\vec{x}_0$. Is the list $\vec{x}_0$, $\vec{v}$, $\vec{w}$ in right hand order or left hand order, or neither? *(Hint: Consider how you can use some of the work from part (a). And be sure to explain your reasoning!)*

Rewriting as $8x - 4y - 6z = -5$, we have

$\vec{n} = (8, -4, -6) = \vec{v} \times \vec{w}$.

Then $-5 = \vec{n} \cdot \vec{x}_0 = \vec{x}_0 \cdot (\vec{v} \times \vec{w})$, which tells us

that $\vec{x}_0, \vec{v}, \vec{w}$ is in left hand order.

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2. *(20 pts)* A particle has initial position $\vec{x}_0 = (1, 2, 3)$ and velocity given by $\vec{v}(t) = (6t^2, 3e^t, 4\sin t)$.

(a) Find an expression for the acceleration of the particle as a function of time.

$$\vec{a} = \vec{v}' = \left(6t^2, 3e^t, 4\sin t\right)'$$
$$= \left(12t, 3e^t, 4\cos t\right)$$

(b) Find an expression for the position of the particle as a function of time.

$$\vec{x} = \int \vec{v}\, dt = \begin{pmatrix} \int 6t^2\, dt \\ \int 3e^t\, dt \\ \int 4\sin t\, dt \end{pmatrix} = \begin{pmatrix} 2t^3 \\ 3e^t \\ -4\cos t \end{pmatrix} + \vec{c}$$

At $t=0$, we have

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \vec{c} \Rightarrow \vec{c} = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$$

So

$$\vec{x} = \begin{pmatrix} 2t^3 \\ 3e^t \\ -4\cos t \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$$

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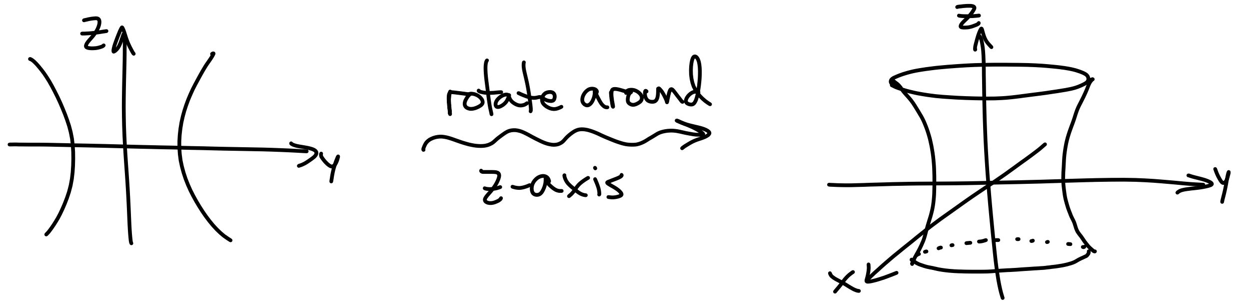
3. *(20 pts)* The surface S has equation $x^2 + (2y-4)^2 - z^2 = 1$. Describe S by identifying

(a) a rotationally symmetric surface R (and explain how you know what R looks like); and

(b) a sequence of geometric transformations that turn R into S.

(a) Let R have equation $x^2+y^2-z^2=1$. This is rotationally symmetric around the z-axis. And in the yz-plane ($x=0$) the equation becomes $y^2-z^2=1$.



So R is this hyperboloid of one sheet.

(b) We can turn this R into S by

$x^2+y^2-z^2=1$

"y" $\mapsto$ "$y-4$" ← (This shifts by 4 in the positive y direction)

$x^2+(y-4)^2-z^2=1$

"y" $\mapsto$ "$2y$" ← (This squishes by a factor of 2 in the y direction)

$x^2+(2y-4)^2-z^2=1$

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4. *(20 pts)*

(a) Find an equation for the graph of the function $f : \mathbb{R}^3 \to \mathbb{R}^1$ defined by $f(v, w, x) = v^2 w - w x^3$.

The graph of f has equation

$$z = f(v,w,x)$$

$$z = v^2 w - w x^3$$

(b) Find a function $h : \mathbb{R}^n \to \mathbb{R}^m$ for which one of the level sets is the sphere with equation $(x+1)^2 + y^2 + z^2 = 9$, and identify the corresponding values of n and m.

This is the $h=9$ level set of the function $h: \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$h(x,y,z) = (x+1)^2 + y^2 + z^2$$

(c) Identify a level set of the function g (defined by $g(x, y) = (y - x^2)^2$) that is also the graph of another function $p : \mathbb{R}^a \to \mathbb{R}^b$, and identify that other function p and the values a and b.

Level sets of g have equation $(y-x^2)^2 = c$, and for $c > 0$ this consists of the two curves,

$$y - x^2 = \sqrt{c} \quad \text{and} \quad y - x^2 = -\sqrt{c}.$$

As such, each level set fails the vertical line test.

But with $c = 0$, we have

$$(y-x^2)^2 = 0 \iff y - x^2 = 0 \iff y = x^2.$$

This is the graph $y = p(x)$ of $p: \mathbb{R}^1 \to \mathbb{R}^1$, $p(x) = x^2$.

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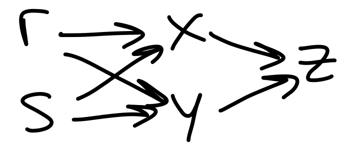
5. *(20 pts)*

(a) We are given that $z = x^2 - y^2$, $x = 3r + 2s$, and $y = r - 5s$. Find an expression for $\dfrac{\partial z}{\partial s}$ in terms of the partials of z with respect to x and y. *Reminder – do NOT plug in to find z explicitly as a function of r and s!*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$



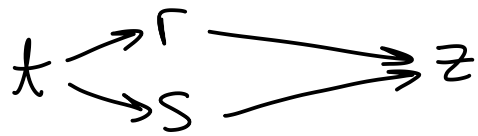
$$= (2x)(2) + (-2y)(-5)$$

$$= 4x + 10y$$

$$= 4(3r+2s) + 10(r-5s) = 22r - 42s$$

(b) Suppose now that $r = 2\cos t$ and $s = 3\sin t$. Compute the value of $\dfrac{dz}{dt}$ when $t = 0$.

(Hint: Evaluate $\dfrac{dr}{dt}(0)$ first.)



$$\frac{dr}{dt} = -2\sin t$$

$$\frac{dr}{dt}(0) = 0$$

$$t=0 \Rightarrow (r,s) = (2,0)$$

$$\frac{dz}{dt}(0) = \frac{\partial z}{\partial r}(2,0)\,\frac{dr}{dt}(0) + \frac{\partial z}{\partial s}(2,0)\,\frac{ds}{dt}(0)$$

Since $\frac{dr}{dt}(0) = 0$ and $\frac{ds}{dt} = 3\cos t$,

$$\frac{dz}{dt}(0) = \frac{\partial z}{\partial s}(2,0)\,\frac{ds}{dt}(0)$$

$$= (44)(3)$$

$$= 132$$

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