EXAM 1

Math 212, 2020 Spring, Clark Bray.

Name: Solutions Section: Student ID:

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

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1. *(27 pts)* The force on a particle P_1 of charge q moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ is given by $\vec{F} = q\vec{v} \times \vec{B}$.

(a) In certain units, a particle has charge times velocity of $q\vec{v} = (4, 1, 2)$ in a magnetic field $\vec{B} = (1, 2, 3)$. Compute the force on this particle.

$$\vec{F} = q\vec{v} \times \vec{B} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -10 \\ 7 \end{pmatrix}$$

(b) Take the derivative of the equation $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$ and recall that $\vec{F} = m\vec{a}$ to show that magnetic force does not change the speed of the particle.

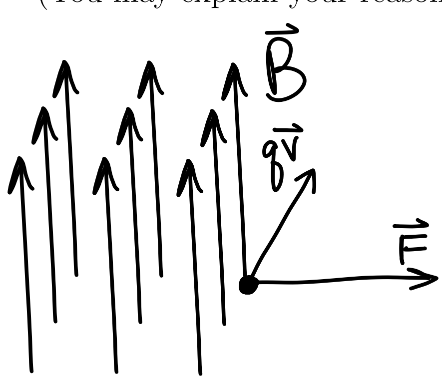
$$2\|\vec{v}\| \frac{d\|\vec{v}\|}{dt} = \frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = 2\vec{v} \cdot \frac{d\vec{v}}{dt} = 2\vec{v} \cdot \vec{a}$$

$\vec{F} = q\vec{v} \times \vec{B}$ is orthogonal to $\vec{v}$, so $\vec{a} = \frac{\vec{F}}{m}$ is also.

So $\|\vec{v}\| \frac{d\|\vec{v}\|}{dt} = \vec{v} \cdot \vec{a} = 0$

and thus $\frac{d\|\vec{v}\|}{dt} = 0$.

(c) If a positively-charged particle P_2 enters a magnetic field that points uniformly upward with a velocity perpendicular to the field, the result is that the particle moves in a horizontal circle. As seen from above, is the motion around that circle clockwise or counterclockwise? (You may explain your reasoning in words and/or pictures.)



By the right hand rule, the force will pull the particle consistently to the right. So the particle will move clockwise as seen from above.

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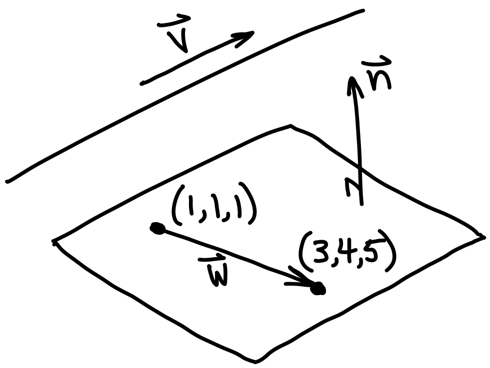
2. *(22 pts)*

(a) Find the equation of the plane P that contains the points $(1,1,1)$ and $(3,4,5)$ and that is parallel to the line parametrized by $\vec{x}(t) = (2-4t, 3t-1, t+2)$.



$$\vec{x}(t) = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}, \text{ so } \vec{v} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}.$$

$$\text{and } \vec{w} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

$\vec{n}$ must be orthogonal to both $\vec{v}, \vec{w}$, so

$$\vec{v} \times \vec{w} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \\ -18 \end{pmatrix}, \parallel \text{ to } \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

So we choose $\vec{n} = (1,2,-2)$, and with $\vec{x}_0 = (1,1,1)$ in P we get

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0 \implies x + 2y - 2z = 1$$

(b) Parametrize the line described by the equations $2x - 6 = 3y + 1 = 4z$.

Set $t = 2x - 6 = 3y + 1 = 4z$.

Then $x = \frac{t+6}{2}$, $y = \frac{t-1}{3}$, $z = \frac{t}{4}$, and

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{t+6}{2} \\ \frac{t-1}{3} \\ \frac{t}{4} \end{pmatrix}$$

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3. *(11 pts)* A particle is moving with acceleration $\vec{a} = (6t, 3\sin t, 5e^t)$, and $\vec{v}(2) = (1, 1, 1)$. Compute the velocity $\vec{v}(t)$.

$$\vec{v} = \int \vec{a}(t)\,dt = \int \begin{pmatrix} 6t \\ 3\sin t \\ 5e^t \end{pmatrix} dt = \begin{pmatrix} 3t^2 \\ -3\cos t \\ 5e^t \end{pmatrix} + \vec{c}$$

At $t=2$ this becomes

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ -3\cos(2) \\ 5e^2 \end{pmatrix} + \vec{c} \implies \vec{c} = \begin{pmatrix} -11 \\ 1+3\cos(2) \\ 1-5e^2 \end{pmatrix}$$

So $\vec{v} = \begin{pmatrix} 3t^2 \\ -3\cos t \\ 5e^t \end{pmatrix} + \begin{pmatrix} -11 \\ 1+3\cos(2) \\ 1-5e^2 \end{pmatrix}$

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4. *(16 pts)*

(a) The surface S with equation $\frac{(x+2)^2}{3} + y^2 - 5z^2 = 1$ can be obtained from the surface M with equation $x^2 + y^2 - z^2 = 1$ by applying the sequence of geometric transformations below.

i. Translate in the x-direction by A;

ii. Stretch in the x-direction by the factor B;

iii. Stretch in the z-direction by the factor C.

Find the values A, B, C.

Rewrite the equation as $\left(\frac{x+2}{\sqrt{3}}\right)^2 + y^2 - (\sqrt{5}\,z)^2 = 1$

$x^2+y^2-z^2=1 \longleftarrow M$

"x" $\mapsto$ "$x+\frac{2}{\sqrt{3}}$", translates by $A = -\frac{2}{\sqrt{3}}$.

$\left(x+\frac{2}{\sqrt{3}}\right)^2 + y^2 - z^2 = 1$

"x" $\mapsto$ "$\frac{x}{\sqrt{3}}$", stretches by $B = \sqrt{3}$.

$\left(\frac{x}{\sqrt{3}}+\frac{2}{\sqrt{3}}\right)^2 + y^2 - z^2 = 1$

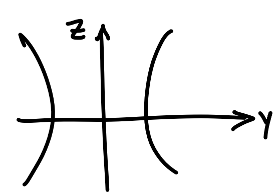
"z" $\mapsto$ "$\sqrt{5}\,z$", stretches by $C = \frac{1}{\sqrt{5}}$.

$\left(\frac{x+2}{\sqrt{3}}\right)^2 + y^2 - 5z^2 = 1 \longleftarrow S$

(b) Identify the surface M.

M is rotationally symmetric around the z-axis.

The cross section in the yz-plane $(x=0)$ is $y^2-z^2=1$:



Rotating this around the z-axis shows M is a hyperboloid of 1 sheet.

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5. *(24 pts)*

(a) The graph of $f(x, y, z) = x^3 - y^2 + z$ is a level set of the function g. Identify the domain and the target of such a function g and give a formula for it.

The graph of f has equation $w = x^3 - y^2 + z$.

This equation is equivalent to $w - x^3 + y^2 - z = 0$,

which is the $g=0$ level set of

$$g : \mathbb{R}^4 \to \mathbb{R}^1, \quad g(x,y,z,w) = w - x^3 + y^2 - z$$

(b) The level set $h = 0$ of the function $h(x, y, z) = 3x - 2y + 4z$ is the graph of a function m. Identify the domain and the target of such a function m and give a formula for it.

The level set $h=0$ has equation $3x - 2y + 4z = 0$.

This equation is equivalent to $z = \dfrac{-3x + 2y}{4}$,

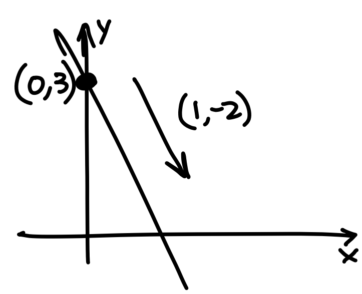
which is the equation of the graph of

$$m : \mathbb{R}^2 \to \mathbb{R}^1, \quad m(x,y) = \frac{-3x + 2y}{4}$$

(c) Parametrize the graph of the function $f(x) = 3 - 2x$.

The equation of the graph is $y = 3 - 2x$.

This passes through $(0,3)$ and is parallel to $(1,-2)$. So a parametrization is $\vec{x}(t) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.



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