EXAM 2

Math 212, 2020 Fall.

Name: Solutions NetID:__________ Student ID:__________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, so that your work will scan as clearly as possible.

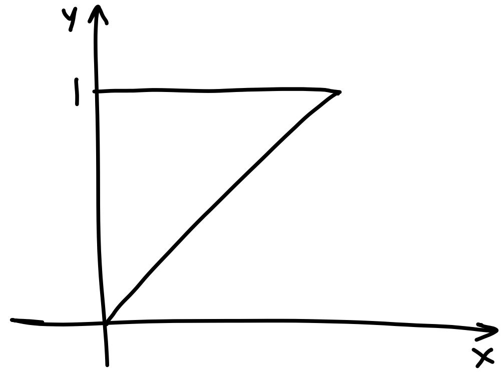
DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

(Scratch space. Nothing on this page will be graded!)

1. *(15 pts)* The domain D in the xy-plane is bounded by the lines $x = 0$, $y = 1$, and $y = x$. Mass is distributed through D with density given by $\delta(x, y) = e^{(1-x)^2} + e^{y^2}$. Compute the total mass in D. *(Hint: Think about the order of the differentials.)*



$$m = \iint \delta \, dA$$

$$= \iint e^{(1-x)^2} + e^{y^2} \, dA$$

$$= \int_0^1 \int_x^1 e^{(1-x)^2} \, dy \, dx + \int_0^1 \int_0^y e^{y^2} \, dx \, dy$$

$$= \int_0^1 (1-x) e^{(1-x)^2} \, dx + \int_0^1 y e^{y^2} \, dy$$

$$= -\frac{1}{2} e^{(1-x)^2} \Big]_0^1 + \frac{1}{2} e^{y^2} \Big]_0^1$$

$$= \frac{1}{2}(e-1) + \frac{1}{2}(e-1) = e-1$$

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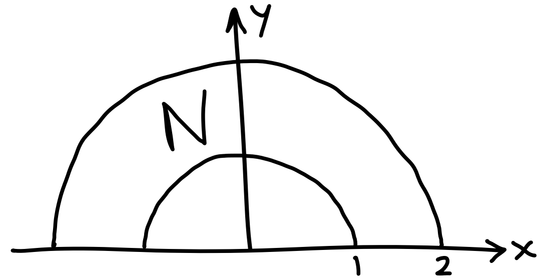
2. *(15 pts)* The region N in the xy-plane is defined by $1 \leq r \leq 2$ and $y \geq 0$. Compute the coordinates $\bar{x}$ and $\bar{y}$ of the centroid of this region. *(Hint: Each of these can be found without directly computing any iterated integrals.)*



$$\bar{x} = \frac{1}{A} \iint_N x \, dA$$

N is symmetric over the y-axis.

Reflection is $R(x,y) = (-x,y)$, so $f(x,y) = x$ is odd:

$$f(R(x,y)) = f(-x,y) = -x = -f(x,y)$$

So the integral (and so also $\bar{x}$) $= 0$ by symmetry.

N has area

$$A = \frac{1}{2}\pi(2)^2 - \frac{1}{2}\pi(1)^2 = \frac{3\pi}{2}$$

Rotating N around the x-axis makes a spherical shell with volume

$$V = \frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1)^3 = \frac{28\pi}{3}$$

Pappus's theorem then gives us

$$V = 2\pi \bar{y} A$$

$$\frac{28\pi}{3} = 2\pi \bar{y} \left(\frac{3\pi}{2}\right) \Rightarrow \bar{y} = \frac{28}{9\pi}$$

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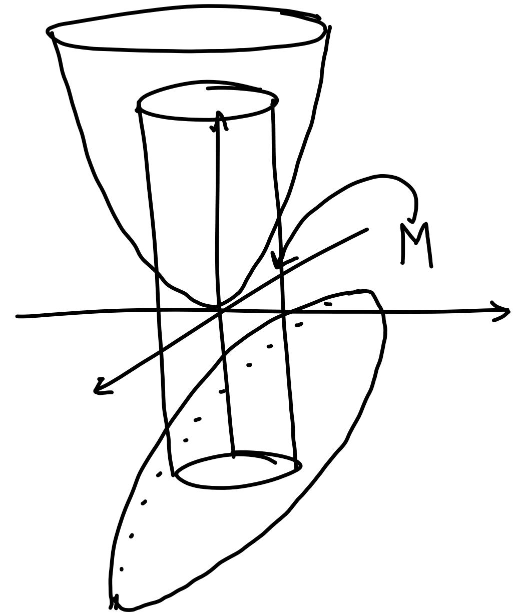
3. *(20 pts)* The solid M is bounded by the surfaces $x^2+y^2=1$, $z=x^2+y^2$, and $z+e^{3x+y^2}=0$.

(a) Write (but do not evaluate yet) an iterated integral representing the integral over M of the function $f(x,y,z)=y^2\sin\left(xyze^{z^2}\right)$.



The projection of M to the xy-plane is the unit disk

$$x^2+y^2=1$$

So we can write

$$\iiint_M f\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-e^{3x+y^2}}^{x^2+y^2} f(x,y,z)\,dz\,dy\,dx$$

(b) Evaluate the integral from part (a) using any methods from this course.

The three bounding surfaces are all symmetric through the xz-plane $\left(R(x,y,z)=(x,-y,z)\right)$, so M is symmetric through this plane too. And f is odd through this plane:

$$f(R(x,y,z)) = f(x,-y,z) = (-y)^2\sin\left(x(-y)ze^{z^2}\right) = y^2\sin\left(-xyze^{z^2}\right)$$
$$= -y^2\sin\left(xyze^{z^2}\right)$$
$$= -f(x,y,z)$$

So $\iiint_M f\,dV = 0$ by symmetry.

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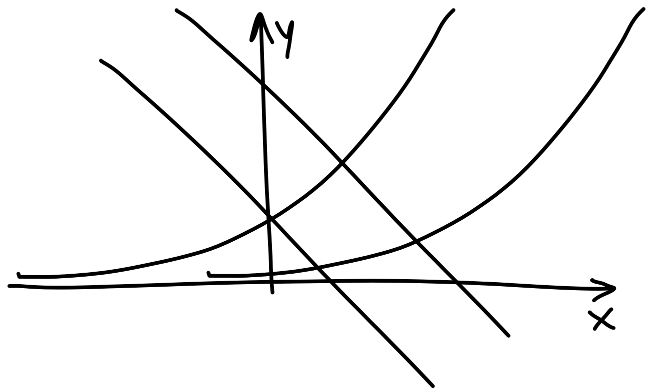
4. *(15 pts)* The region R in the xy-plane is bounded by the curves parametrized by (t, e^t), $(t+2, e^t)$, $(t, 1-t)$, and $(t, 3-t)$. Compute the double integral over R of $g(x, y) = 1 + \frac{1}{y}$.

The curves have equations

$$ye^{-x} = 1 \qquad x+y = 1$$
$$ye^{-x} = e^{-2} \qquad x+y = 3$$



We choose $u = ye^{-x}$, $v = x+y$, and compute

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} -ye^{-x} & e^{-x} \\ 1 & 1 \end{pmatrix} = -(1+y)e^{-x} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{-1}{(1+y)e^{-x}}$$

Then

$$\iint_R \left(1+\frac{1}{y}\right) dx\,dy = \iint \left(\frac{y+1}{y}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint \left(\frac{y+1}{y}\right)\left(\frac{1}{(y+1)e^{-x}}\right) du\,dv$$

$$= \int_1^3 \int_{e^{-2}}^1 \frac{1}{u}\,du\,dv$$

$$= \int_1^3 \left(\ln|u|\Big]_{u=e^{-2}}^{u=1}\right) dv$$

$$= \int_1^3 (0-(-2))\,dv = 4$$

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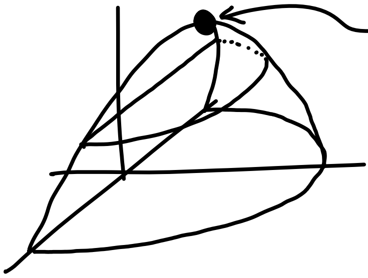
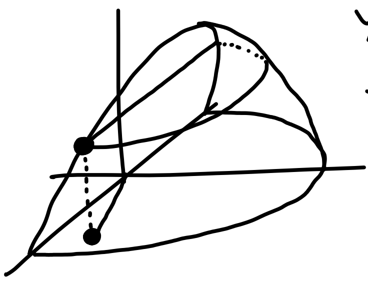
5. *(20 pts)* The region K is the portion of the unit ball (centered at $\vec{0}$) above the xy-plane and below the plane $y = z$.

(a) Write (but do not evaluate) an iterated integral in cylindrical coordinates that represents $\iiint_K xyz\, dV$.

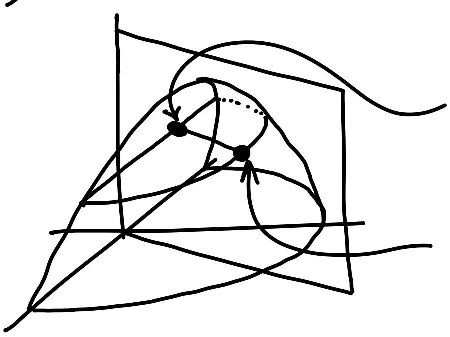


$$\left.\begin{array}{l} x=0,\ y=z, \\ x^2+y^2+z^2=1 \end{array}\right\} \Rightarrow z=\frac{1}{\sqrt{2}} \qquad \text{So } z \in \left[0, \frac{1}{\sqrt{2}}\right]$$



$$\left.\begin{array}{l} y=z,\ x^2+y^2+z^2=1 \\ \Rightarrow x=\sqrt{1-2z^2} \end{array}\right\} \Rightarrow \theta_1 = \arctan\left(\frac{z}{\sqrt{1-2z^2}}\right)$$

$$\text{So } \theta \in \left[\arctan\left(\frac{z}{\sqrt{1-2z^2}}\right), \pi - \arctan\left(\frac{z}{\sqrt{1-2z^2}}\right)\right]$$



$y=z,\ y=r\sin\theta$

$\Rightarrow r_1 = z\csc\theta$

$r^2+z^2=1$

$\Rightarrow r_2 = \sqrt{1-z^2}$

$$\text{So } r \in \left[z\csc\theta, \sqrt{1-z^2}\right]$$

$$\text{So } \iiint_K xyz\, dV = \iiint_K (r\cos\theta)(r\sin\theta)z\,(r\,dr\,d\theta\,dz)$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \int_{\arctan\left(\frac{z}{\sqrt{1-2z^2}}\right)}^{\pi - \arctan\left(\frac{z}{\sqrt{1-2z^2}}\right)} \int_{z\csc\theta}^{\sqrt{1-z^2}} r^3 z \cos\theta \sin\theta \, dr\, d\theta\, dz$$

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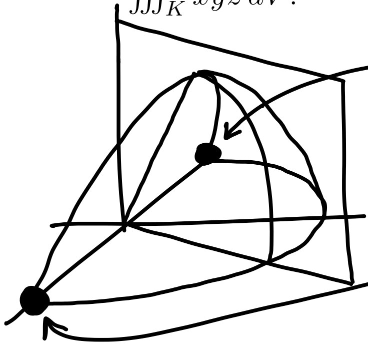
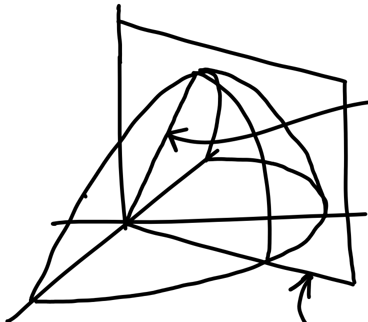
(b) Write (but do not evaluate) an iterated integral in spherical coordinates that represents $\iiint_K xyz\,dV$.



$\theta \in [0, \pi]$



$y = z \Rightarrow \rho \sin\phi \sin\theta = \rho\cos\phi \Rightarrow \phi = \arctan(\csc\theta)$

So $\phi \in \left[\arctan(\csc\theta), \frac{\pi}{2}\right]$

So $\iiint_K xyz\,dV = \iiint_K (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta)$

$$= \int_0^{\pi}\int_{\arctan(\csc\theta)}^{\pi/2}\int_0^1 \rho^5\sin^3\phi\cos\phi\cos\theta\sin\theta\,d\rho\,d\phi\,d\theta$$

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6. *(15 pts)* Write (but do not evaluate) an iterated integral that represents the area of the surface defined by the spherical equation $\rho = \phi$ for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{pmatrix} = \begin{pmatrix} \phi \sin\phi \cos\theta \\ \phi \sin\phi \sin\theta \\ \phi \cos\phi \end{pmatrix}$$

$$\vec{X}_\phi = \begin{pmatrix} (\sin\phi + \phi\cos\phi)\cos\theta \\ (\sin\phi + \phi\cos\phi)\sin\theta \\ (\cos\phi - \phi\sin\phi) \end{pmatrix} \qquad \vec{X}_\theta = \begin{pmatrix} -\phi\sin\phi\sin\theta \\ \phi\sin\phi\cos\theta \\ 0 \end{pmatrix}$$

$$\vec{N} = \vec{X}_\phi \times \vec{X}_\theta = \begin{pmatrix} (\phi^2\sin^2\phi - \phi\sin\phi\cos\phi)\cos\theta \\ (\phi^2\sin^2\phi - \phi\sin\phi\cos\phi)\sin\theta \\ (\phi\sin^2\phi + \phi^2\sin\phi\cos\phi) \end{pmatrix}$$

$$\|\vec{N}\| = \sqrt{(\phi^2\sin^2\phi - \phi\sin\phi\cos\phi)^2 + (\phi\sin^2\phi + \phi^2\sin\phi\cos\phi)^2}$$

$$= \phi\sin\phi\sqrt{(\phi\sin\phi - \cos\phi)^2 + (\sin\phi + \phi\cos\phi)^2}$$

$$= \phi\sin\phi\sqrt{1+\phi^2}$$

$$\text{area} = \int_0^{2\pi}\int_0^{\pi} \phi\sin\phi\sqrt{1+\phi^2}\, d\phi\, d\theta$$

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