EXAM 2

Math 212, 2021 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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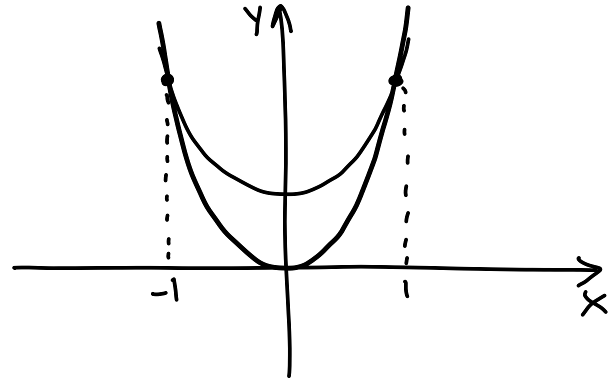
1. *(20 pts)* The region D is bounded by $y = 2x^2$ and $y = 1 + x^2$, and mass is distributed across D as indicated by the density $\delta(x, y) = e^{x^2} + y^{13}$.

(a) Write iterated integrals representing the coordinates of the centroid of this mass (leaving the mass written as "m").



$$\bar{x} = \frac{1}{m}\iint_D x \, dm$$

$$= \frac{1}{m}\int_{-1}^{1}\int_{2x^2}^{1+x^2} (x)\left(e^{x^2} + y^{13}\right) dy\, dx$$

$$\bar{y} = \frac{1}{m}\iint_D y \, dm$$

$$= \frac{1}{m}\int_{-1}^{1}\int_{2x^2}^{1+x^2} (y)\left(e^{x^2} + y^{13}\right) dy\, dx$$

(b) Evaluate ONE of the integrals above using any methods from this course.

The domain for the above integrals is symmetric over the y-axis, with $R(x,y) = (-x, y)$. The integrand $f(x,y) = (x)\left(e^{x^2} + y^{13}\right)$ has odd symmetry over this line because

$$f(R(x,y)) = f(-x, y) = (-x)\left(e^{(-x)^2} + y^{13}\right) = -(x)\left(e^{x^2} + y^{13}\right)$$
$$= -f(x,y)$$

So this integral (and thus $\bar{x}$) is zero by symmetry.

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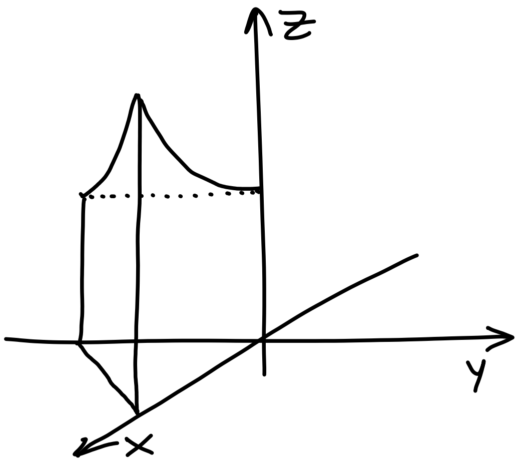
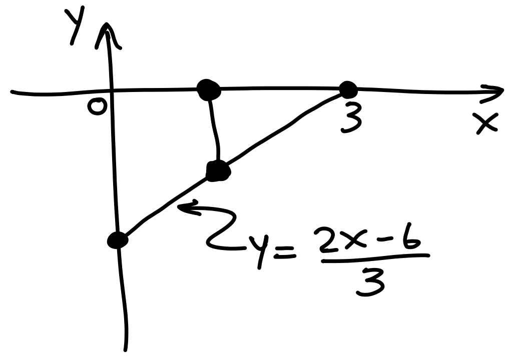
2. *(20 pts)* The solid R is bounded by the three coordinate planes and the surfaces with equations $2x - 3y = 6$ and $z = 1 + x^2$. Write out a *single* triple iterated integral representing $\iiint_R e^z \, dV$. (You do not have to evaluate the integral.)



Projection to xy-plane:



$$\iiint_R e^z \, dV = \int_0^3 \int_{\frac{2x-6}{3}}^{0} \int_0^{1+x^2} e^z \, dz \, dy \, dx$$

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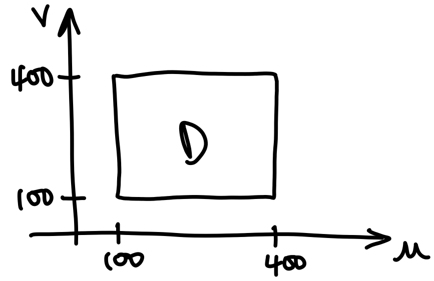
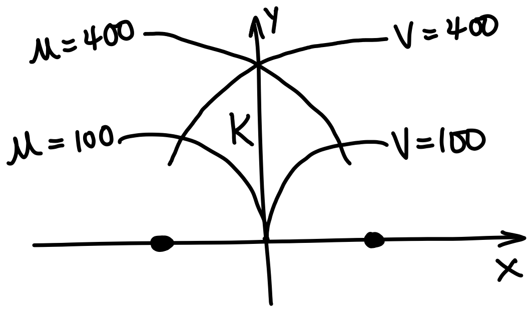
3. *(20 pts)* The region K in the xy-plane is the collection of points above the x-axis whose distances to $(-10, 0)$ and $(10, 0)$ are both between 10 and 20. Compute $\iint_K y\,dx\,dy$. *(Hint: Let u and v be the squares of the relevant distances.)*



$$u = (x+10)^2 + y^2$$
$$v = (x-10)^2 + y^2$$



$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 2(x+10) & 2y \\ 2(x-10) & 2y \end{pmatrix} = 80y \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{80y}$$

$$\iint_K y\,dx\,dy = \iint_D (y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_D \frac{1}{80}\,du\,dv$$

$$= \frac{(\text{area of } D)}{80} = \frac{300^2}{80} = \frac{90{,}000}{80} = 1125$$

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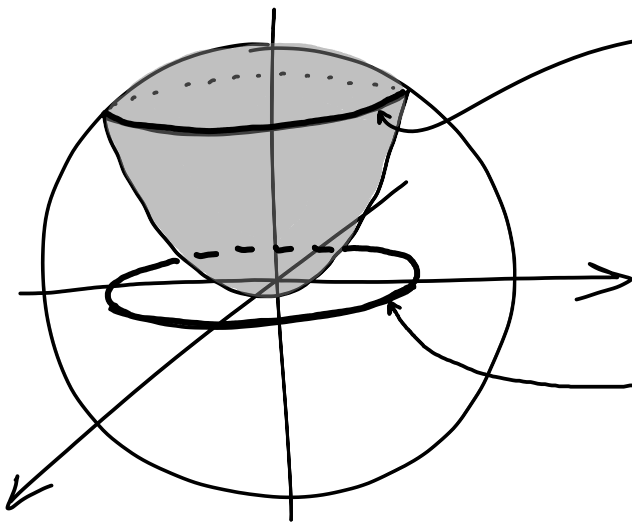
4. *(20 pts)* The solid T is defined by $x^2 + y^2 + z^2 \leq 2$ and $z \geq x^2 + y^2$. Compute the volume of T.



$x^2+y^2+z^2=2$ <u>and</u> $z=x^2+y^2$

$\Rightarrow z^2+z-2=0$

$(z+2)(z-1)=0$

$\Rightarrow z=1$

$\Rightarrow x^2+y^2=1$

$$V = \iiint dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\sqrt{2-r^2} - r^3 \, dr\,d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{3}(2-r^2)^{3/2} - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{2\sqrt{2}-1}{3} - \frac{1}{4}\, d\theta$$

$$= (2\pi)\left(\frac{8\sqrt{2}-4-3}{12}\right) = \frac{8\sqrt{2}-7}{6}\pi$$

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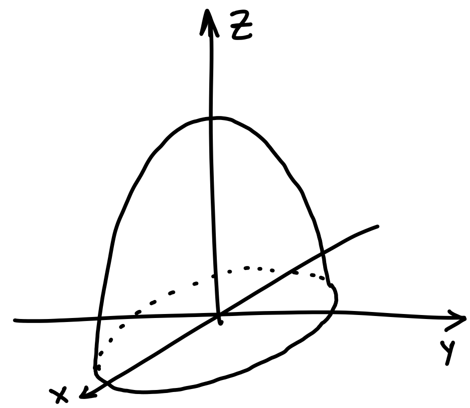
5. *(20 pts)* The surface S is the part of $z = 4 - x^2 - 4y^2$ that is above the xy-plane. Ants are crawling on S with the number of ants per unit area given by $\delta(x, y, z) = z^2$. Write out (but do not evaluate) a *single* iterated integral representing the total number of ants on S.

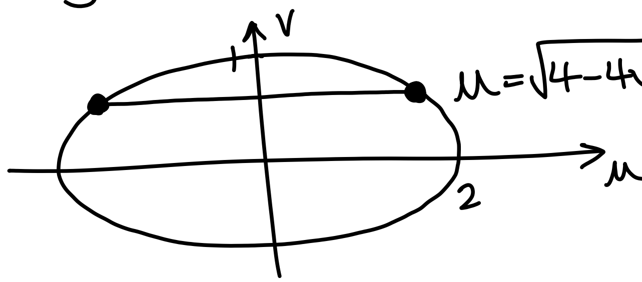
We parametrize S by

$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ 4-u^2-4v^2 \end{pmatrix}$$



$$\vec{X}_u = \begin{pmatrix} 1 \\ 0 \\ -2u \end{pmatrix} \quad \vec{X}_v = \begin{pmatrix} 0 \\ 1 \\ -8v \end{pmatrix} \quad \vec{N} = \begin{pmatrix} 2u \\ 8v \\ 1 \end{pmatrix} \quad \|\vec{N}\| = \sqrt{1+4u^2+64v^2}$$

u and v range over $u^2+4v^2 \le 4$:



$u = \sqrt{4-4v^2} = 2\sqrt{1-v^2}$

So we have

$$a = \iint_S \delta \, dS = \iint z^2 \|\vec{N}\| \, du \, dv$$

$$= \int_{-1}^{1} \int_{-2\sqrt{1-v^2}}^{2\sqrt{1-v^2}} (4-u^2-4v^2)^2 \sqrt{1+4u^2+64v^2} \, du \, dv$$

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